IOP Publishing European Journal of Physics
Eur. J. Phys. **37** (2016) 055005 (4pp) doi:10.1088/0143-0807/37/5/055005

The rain-powered cart

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Received 10 May 2016, revised 22 June 2016
Accepted for publication 11 July 2016
Published 5 August 2016

Abstract

A frictionless cart in the shape of a right triangle (with the vertical side forward) is elastically impacted by vertically falling raindrops. The speed of the cart as a function of time can be analytically deduced as an exercise in the use of trigonometric and hyperbolic functions. A characteristic time defines the approach to a terminal speed which is a sizeable fraction of the speed of the rain. The treatment is accessible to a student in a calculus-based mechanics course.

Keywords: elastic collisions, rainfall, terminal speed, renewable energy

(Some figures may appear in colour only in the online journal)

A familiar problem treats how wet a person walking in rain becomes as they travel a given horizontal distance at different speeds [1–3]. As a variation on this scenario, consider a triangular cart that can roll along a horizontal surface under the impulse of raindrops (or hail) falling vertically at terminal speed u that bounce elastically [4, 5] off the cart's two surfaces sketched in figure 1. Its front vertical surface has area A, whereas its trailing surface has area B and is inclined at angle θ to the horizontal so that

$$A = B \sin \theta. \qquad (1)$$

Rolling friction [6] and air drag are neglected. The cart starts from rest so that $v(0) = 0$.

To calculate the force that the rain exerts on the two surfaces, jump into the frame of reference of the cart [7]. The rain, of volumetric mass density ρ which is some fraction of $1000\ \mathrm{kg\ m^{-3}}$, is then moving downward at speed u and rightward at speed v *from the point of view of the cart*. The component of the rain's velocity normal to the front surface is v so that the force it exerts on that surface is

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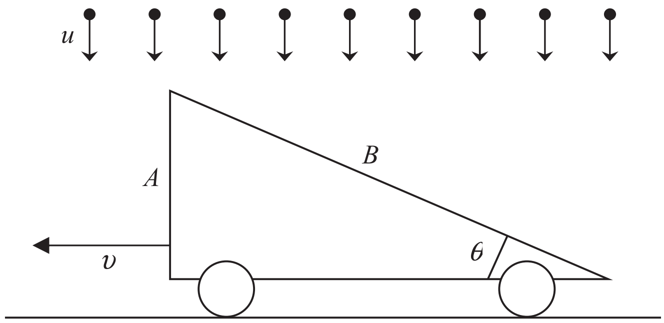

Figure 1. Motion of the cart in the laboratory frame of reference.

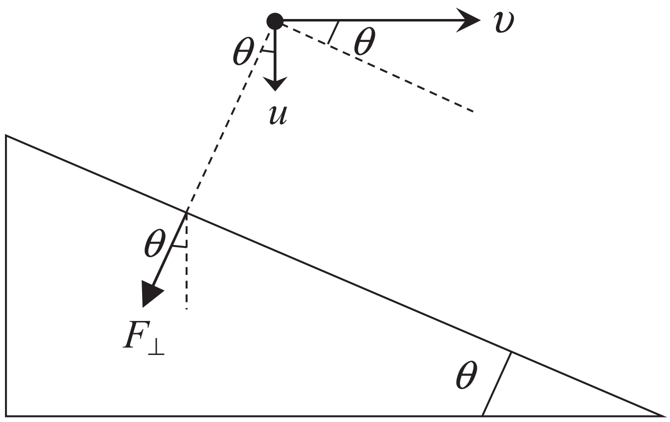

Figure 2. Motion of the rain in the cart's frame of reference.

$$F_{\text{front}} = -(\rho A \upsilon)(2\upsilon), \tag{2}$$

where the minus sign indicates this force is backward relative to the direction of travel of the cart, the contents of the first set of parentheses are the rate at which mass hits the front surface, and the contents of the second set are the velocity change of the raindrops. The factor of 2 arises from the fact that the collisions are elastic rather than inelastic [8]. In like fashion, the normal component of the force on the sloped portion of the cart's surface is

$$F_\perp = 2\rho B u_\perp^2, \tag{3}$$

where the component of the rain's velocity perpendicular to the back surface of area B in the cart's frame of reference is

$$u_\perp = u \cos\theta - \upsilon \sin\theta \tag{4}$$

according to figure 2. The horizontal component of this force is thus

$$F_{\text{back}} = +F_\perp \sin\theta = 2\rho B (u \cos\theta - \upsilon \sin\theta)^2 \sin\theta \tag{5}$$

which is the force pushing the cart forward. Substituting equation (1) into (5) and adding that to equation (2) gives the net force on the cart. Thus if the cart has mass M, Newton's second law becomes

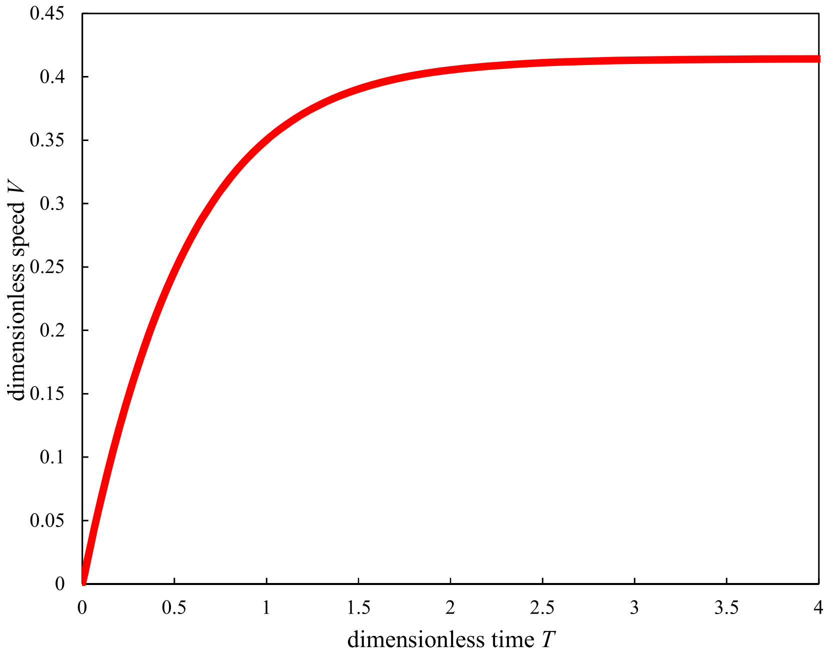

Figure 3. Speed of the cart starting from rest and approaching terminal speed for an incline angle θ of its rear surface of 45°.

$$\begin{aligned} M\frac{\mathrm{d}v}{\mathrm{d}t} &= 2\rho A[(u\cos\theta - v\sin\theta)^2 - v^2] \\ &= 2\rho A[u^2\cos^2\theta + v^2(\sin^2\theta - 1) - 2uv\cos\theta\sin\theta]. \end{aligned} \tag{6}$$

Introducing the dimensionless variables $V \equiv v/u$ and $T \equiv 2\rho AuM^{-1}t\cos\theta$, this equation can be separated and integrated as

$$\begin{aligned} T\cos\theta &= \int_0^V \frac{\mathrm{d}V}{1 - V^2 - 2V\tan\theta} \\ &= \int_0^V \frac{\mathrm{d}V}{(1 + \tan^2\theta) - (V + \tan\theta)^2}. \end{aligned} \tag{7}$$

To do this integral, change variable from V to x where $V + \tan\theta = \sec\theta\tanh x$ to get

$$T\cos\theta = \int_{\tanh^{-1}(\sin\theta)}^{\tanh^{-1}(V\cos\theta + \sin\theta)} \frac{\sec\theta\,\mathrm{sech}^2 x\,\mathrm{d}x}{\sec^2\theta\,\mathrm{sech}^2 x} \tag{8}$$

so that

$$T = \tanh^{-1}(V\cos\theta + \sin\theta) - \tanh^{-1}(\sin\theta) \tag{9}$$

gives the dimensionless time T required to attain some final dimensionless cart speed V. This equation can be inverted using the hyperbolic identity $\tanh(\alpha + \beta) = (\tanh\alpha + \tanh\beta)/(1 + \tanh\alpha\tanh\beta)$ as

$$\frac{\tanh T + \sin\theta}{1 + \sin\theta\tanh T} = V\cos\theta + \sin\theta \tag{10}$$

so that

$$v(t) = \frac{u \cos\theta}{\sin\theta + \coth(2\rho A u M^{-1} t \cos\theta)} \tag{11}$$

after restoring dimensional quantities. This equation correctly predicts that $v = 0$ if $u = 0$, $\rho = 0$, $A = 0$ (which implies $\theta = 0$ according to figure 1), $t = 0$, or $M \to \infty$.

Assuming that $\rho A u M^{-1} \cos\theta$ is nonzero, then the terminal speed of the cart is found from equation (11) to be

$$v_\infty = \frac{u \cos\theta}{\sin\theta + 1} \tag{12}$$

which can alternatively be deduced by setting the right-hand side of the first line in equation (6) to zero (since the cart's acceleration $\mathrm{d}v/\mathrm{d}t = 0$ at terminal speed). The terminal speed monotonically decreases with increasing θ. Specifically, the value of v_∞/u is approximately 1 for small θ, $\sqrt{2} - 1 \approx 0.41$ for $\theta = 45°$ as plotted in figure 3, and 0 for θ approaching 90°.

To give a feel for the numerics, assume a quite heavy rainfall having a terminal speed of $u = 5\ \mathrm{m\ s^{-1}}$ [9] and a ground gauge collection rate of $r = 7\ \mu\mathrm{m\ s^{-1}}$ (corresponding to 1 inch per hour) with $\rho/\rho_{\mathrm{water}} = r/u$ where $\rho_{\mathrm{water}} = 1000\ \mathrm{kg\ m^{-3}}$. For a cart of average specific gravity $s = \rho_{\mathrm{cart}}/\rho_{\mathrm{water}} = 0.5$, the ratio of its mass to its front surface area is $M/A = \rho_{\mathrm{cart}} L/2$ where the length of its base is say $L = 0.2$ m, neglecting the volume of the wheels. For $\theta = 45°$, one unit along the horizontal axis in figure 3 then corresponds to a characteristic time of

$$\frac{M}{2\rho A u \cos\theta} = \frac{sL}{2r\sqrt{2}} \tag{13}$$

or 84 min in which the cart reaches 85% of its terminal speed of $v_\infty = 2.1\ \mathrm{m\ s^{-1}}$ (corresponding to $7.5\ \mathrm{km\ h^{-1}}$), powered by the renewable energy of rain [10, 11].

References

[1] Bocci F 2012 Whether or not to run in the rain *Eur. J. Phys.* **33** 1321–32
[2] Banks R B 1999 *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics* (Princeton, NJ: Princeton University Press) ch 12
[3] http://discovery.com/tv-shows/mythbusters/videos/running-in-the-rain-minimyth/
[4] de Izarra C 2012 Duration of an elastic collision *Eur. J. Phys.* **33** 997–1006
[5] Gilet T and Bush J W M 2012 Droplets bouncing on a wet, inclined surface *Phys. Fluids* **24** 122103
[6] Mungan C E 2012 Rolling friction on a wheeled laboratory cart *Phys. Educ.* **47** 288–92
[7] Funk D 2006 Elastic collisions with two moving objects made simple *Phys. Teach.* **44** 80–1
[8] Featonby D 2016 Decisions balls—the answer *Phys. Educ.* **51** 027001
[9] Beard K V 1976 Terminal velocity and shape of cloud and precipitation drops aloft *J. Atmos. Sci.* **33** 851–61
[10] Perera K C R, Sampath B G, Dassanayake V P C and Hapuwatte B M 2014 Harvesting of kinetic energy of the raindrops *Int. J. Math., Computat., Phys., Elec. Comput. Engin.* **8** 325–30
[11] Martin S and Sharma A K 2014 Analysis on rainwater harvesting and its utilization for pico hydro power generation *Int. J. Adv. Res. Comput. Engin. Technol.* **3** 2121–6